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1999 J. Phys. A: Math. Gen. 32 5251

(http://iopscience.iop.org/0305-4470/32/28/305)

J. Phys. A: Math. Gen. **32** (1999) 5251–5260. Printed in the UK PII: S0305-4470(99)01401-8

Langevin equations coupled through correlated noises

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Received 27 January 1999, in final form 6 May 1999

Abstract. We consider the dynamics of non-interacting Brownian particles which are driven by correlated (non-independent) noise sources. In simple confining potentials the particles tend to aggregate as the noise correlation is increased. If two particles are subject to the same noise they will coalesce and remain together ever after. We show that complete aggregation of the particles can be expected even in the case of a disordered potential which does not confine the individual particle trajectories. Finally, we examine the case of correlation in the noises which depends on the separation of the two particles.

An important property of driven dissipative systems is their ability to form ordered structures under non-equilibrium conditions [1]. There are many mechanisms responsible for creating correlations, and much effort, both experimental and theoretical, has been invested in the study of the resulting non-equilibrium steady states. Systems of this type include chemical reactions and reaction–diffusion fronts [2], driven lattice gases [3], phase-ordering kinetics [4] and clustering in granular media [5]. In all these examples, inter-particle interactions give rise to correlated dynamics.

In this paper we will discuss an alternative mechanism which can produce non-trivial, ordered steady states. We consider the behaviour of particles which do not interact directly, but are driven by correlated noise sources [6]. We will demonstrate that aggregation of the particles results from an interplay between the noise correlation and the common environment in which the particles move. Physical realizations of such a mechanism arise in many contexts as, for example, in shaken granular media [5]. Furthermore, correlated noises of the type considered here has wider applications, ranging from the modelling of communal behaviour in biological systems [7] to investment strategies in the stock market [8]. Similar problems have also been investigated in relation to the control of chaotic dynamical systems [9–11], a problem which finds potential applications in, for instance, secure communication [12].

We will first introduce a simple two-particle model which will be used to illustrate the aggregation phenomenon and define the relevant parameters. Next we consider more general confining potentials and show numerically how order develops in these situations. We will then investigate noise-correlated dynamics in random environments and show that, even in cases where the individual particle trajectories are not localized in space, correlated noises can

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induce aggregation. We conclude by studying the Brownian motion of two particles whose noise correlation strength is a function of their separation. We will argue that depending on the form of the correlation, the particles can either aggregate or remain apart forever.

The motion of a particle in a highly viscous medium can be described by a Langevin equation of the form

$$\dot{x}(t) = -\frac{\partial V}{\partial x} + \eta(t) \tag{1}$$

where $V(x)$ is the external potential, $\eta(t)$ is the 'thermal' noise and the acceleration term proportional to $\ddot{x}(t)$ has been considered negligible. The noise is generally assumed to be Gaussian with

$$\begin{aligned} \langle \eta(t) \rangle &= 0 \\ \langle \eta(t)\eta(t') \rangle &= 2D\delta(t - t') \end{aligned} \tag{2}$$

and the stationary state, if it exists, corresponds to a Boltzmann distribution

$$P_{\mathrm{eq}}(x) \propto \mathrm{e}^{-V(x)/D}. \tag{3}$$

Consider now two non-interacting Brownian particles moving in the same external potential $V(x)$, so that the equations of motion are

$$\dot{x}_1(t) = f(x_1) + \eta_1(t) \tag{4}$$

$$\dot{x}_2(t) = f(x_2) + \eta_2(t) \tag{5}$$

where $f(x) = -\frac{\partial V}{\partial x}$ is the force acting on the particles; $\eta_1(t)$ and $\eta_2(t)$ are Gaussian noises which satisfy

$$\langle \eta_i(t) \rangle = 0 \qquad i = 1, 2 \tag{6}$$

and

$$\langle \eta_i(t)\eta_j(t') \rangle = 2\gamma_{ij}\delta(t - t') \qquad i, j = 1, 2. \tag{7}$$

The motion of the individual particles is coupled through the correlation matrix γ_{ij}, which we will assume to have the following form:

$$\gamma_{ij} = D \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}. \tag{8}$$

If $\epsilon = 0$ then the two particles move independently and the steady state will be given by the product of two single-particle distribution functions, equation (3). On the other hand, if $\epsilon = 1$ the stationary distribution is formally [6]

$$P_{\mathrm{eq}}(x_1, x_2) \propto \delta(x_1 - x_2)\mathrm{e}^{-V(x_1)/D}. \tag{9}$$

In passing, it is interesting to note that this solution remains valid even if a repulsive force between the two particles is introduced [10]. Here we are interested in the case $0 < \epsilon < 1$ and ask the following question: how does the steady state of the two particles depend on the noise correlation, and, in particular, can spatial correlations in the noise induce dynamical correlations in the relative motion of the particles? In the following investigation we will limit ourselves to one-dimensional systems. For confining $V(x)$, however, equations (3) and (9) hold in any dimensions, possibly suggesting that in this case there is no effect related to the dimensionality of the space.

In order to gain some insight into the problem we first consider the simple situation of two particles confined by a harmonic potential and coupled through the noise correlator equation (8). In this case $V(x) = \frac{1}{2}kx^2$ and the equations of motion become

$$\dot{x}_1(t) = -kx_1 + \eta_1(t) \tag{10}$$

$$\dot{x}_2(t) = -kx_2 + \eta_2(t). \tag{11}$$

With the change of variables

$$s = x_1 + x_2 \tag{12}$$

$$d = x_1 - x_2 \tag{13}$$

the equations of motion decouple and the steady-state probability density $P(x_1, x_2)$ can be determined exactly for all ϵ. The resulting solution is

$$P(x_1, x_2) = \frac{k}{2\pi(1-\epsilon^2)^{1/2}} \mathrm{e}^{-\frac{k}{2D}\frac{x_1^2}{1+\epsilon}} \mathrm{e}^{-\frac{k}{2D}\frac{x_2^2}{1+\epsilon}} \mathrm{e}^{-\frac{k}{2D}(x_1-x_2)^2\frac{\epsilon}{1-\epsilon^2}}. \tag{14}$$

In the limit $\epsilon \to 0$ we recover the expected product distribution whilst for $\epsilon > 0$, equation (14) indicates that the particles tend to cluster; the joint probability distribution is peaked around $x_1 = x_2$. Furthermore, in the limit $\epsilon \to 1$, one finds (see also [13])

$$P(x_1, x_2) \sim \delta(x_1 - x_2)\mathrm{e}^{-\frac{k}{2D}x_1^2}. \tag{15}$$

Thus, the particles will tend to aggregate and, as they both experience the same noise, once they meet they will remain together ever after. From this simple example it is clear that noise correlation can induce spatial aggregation. It is interesting to remark that an analogous effect arises even by including the acceleration term in the equations of motion (10) and (11). Indeed, in the case of an external harmonic potential, the stationary probability distribution can be worked out exactly and results in complete aggregation in the limit $\epsilon \to 1$.

Next we shall consider the two particles confined by a double-well potential. While such a situation is not amenable to a complete analytic treatment it is readily simulated numerically. We consider a potential of the form $V(x) = x^4/4 - x^2/2$ and measure the probability distribution, in the steady state, for the separation of the particles d, equation (13). Figure 1 shows $P(d)$ for increasing values of ϵ. One sees that the particles tend to aggregate

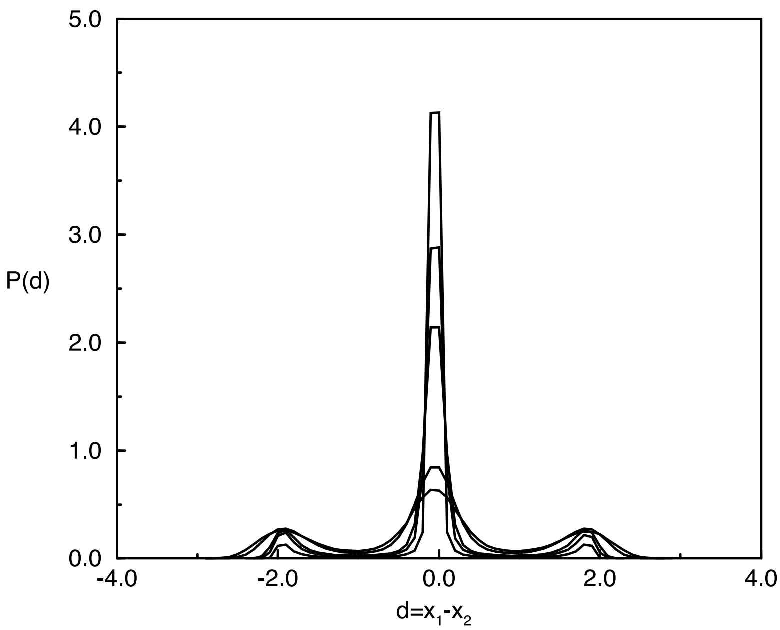

Figure 1. Probability distribution $P(d)$ in the case of a double-well potential. The curves correspond to $\epsilon = 0.2, 0.5, 0.9, 0.95, 0.99$, the larger values of ϵ having more weight at the origin.

as ϵ is increased, and that in the limit $\epsilon \to 1$, one again finds that $P(d)$ tends to $\delta(d)$. Note that in this limit $P(d)$ contains no information of the underlying potential.

As a third example, we consider two particles moving in a infinite potential well

$$V(x) = \begin{cases} 0 & \text{if } 0 < x < L \\ \infty & \text{otherwise.} \end{cases} \tag{16}$$

We will define the dynamics in terms of a discrete space-time master equation, which can be readily solved numerically and used as a starting point to deduce the correct continuum equation of motion and boundary conditions. The coordinates of the particles can take only discrete multiples of the lattice spacing a, and it is convenient to represent the problem on a two-dimensional square lattice. Each point of the lattice $\vec{x} = (x_1, x_2)$ describes the coordinates of the two particles. Let $P(\vec{x}, t)$ be the probability that particle 1 is at x_1 and particle 2 is at x_2, at time t, and $W_{\vec{x},\vec{y}}$ the transition rate from site $\vec{y}$ to site $\vec{x}$ ($\sum_{\vec{x}} W_{\vec{x},\vec{y}} = 1$). In general

$$P(\vec{x}, t + \Delta t) = \sum_{\vec{y}} W_{\vec{x},\vec{y}} P(\vec{y}, t). \tag{17}$$

The microscopic dynamics of a single particle is defined as follows. The particle moves to the left or to the right with equal probability; if it is on a boundary of the well (e.g. $x_1 = L$) and tries to move outside the well (e.g. right) it stays at the same site for that time step. The correlation in the noise can be implemented in the following way: away from the boundaries both particles move in the same direction with a rate $\frac{1+\epsilon}{4}$ while they move in opposite directions with rate $\frac{1-\epsilon}{4}$. If one of the particles is on the boundary and tries to move out of the region it remains on the boundary for that time step. A summary of the transition rates is shown in figure 2. Note that for $\epsilon = 0$ the motion is uncorrelated whereas, in the limit $\epsilon \to 1$, the motion becomes completely correlated, as both particles will move in the same direction unless they are restrained by the boundaries. In the latter case, therefore, the relative distance of the two particles cannot increase and, in particular, it decreases if and only if one of the particles hits the wall.

In terms of these transition rates, the master equation can be written as

$$\begin{aligned} P(x_1, x_2, t + \Delta t) = {} & P(x_1 + a, x_2 + a, t)\frac{1+\epsilon}{4} + P(x_1 - a, x_2 - a, t)\frac{1+\epsilon}{4} \\ & + P(x_1 - a, x_2 + a, t)\frac{1-\epsilon}{4} + P(x_1 + a, x_2 - a, t)\frac{1-\epsilon}{4} \end{aligned} \tag{18}$$

while on a boundary, for example $x_1 = L$,

$$\begin{aligned} P(L, x_2, t + \Delta t) = {} & P(L, x_2 + a, t)\frac{1-\epsilon}{4} + P(L - a, x_2 - a, t)\frac{1+\epsilon}{4} \\ & + P(L - a, x_2 + a, t)\frac{1-\epsilon}{4} + P(L, x_2 - a, t)\frac{1+\epsilon}{4} \end{aligned} \tag{19}$$

and at a corner, for example $x_1 = x_2 = L$,

$$\begin{aligned} P(L, L, t + \Delta t) = {} & P(L, L, t)\frac{1+\epsilon}{4} + P(L - a, L - a, t)\frac{1+\epsilon}{4} \\ & + P(L - a, L, t)\frac{1+\epsilon}{4} + P(L, L - a, t)\frac{1+\epsilon}{4}. \end{aligned} \tag{20}$$

These equations are readily solved numerically and results for the steady-state probability are shown in figure 3. The darker regions of the plot correspond to higher values of the joint probability distribution. Again there is a tendency for the particles to aggregate and figure 4 shows the probability distribution for the particle separation, d. As $\epsilon \to 1$, this distribution

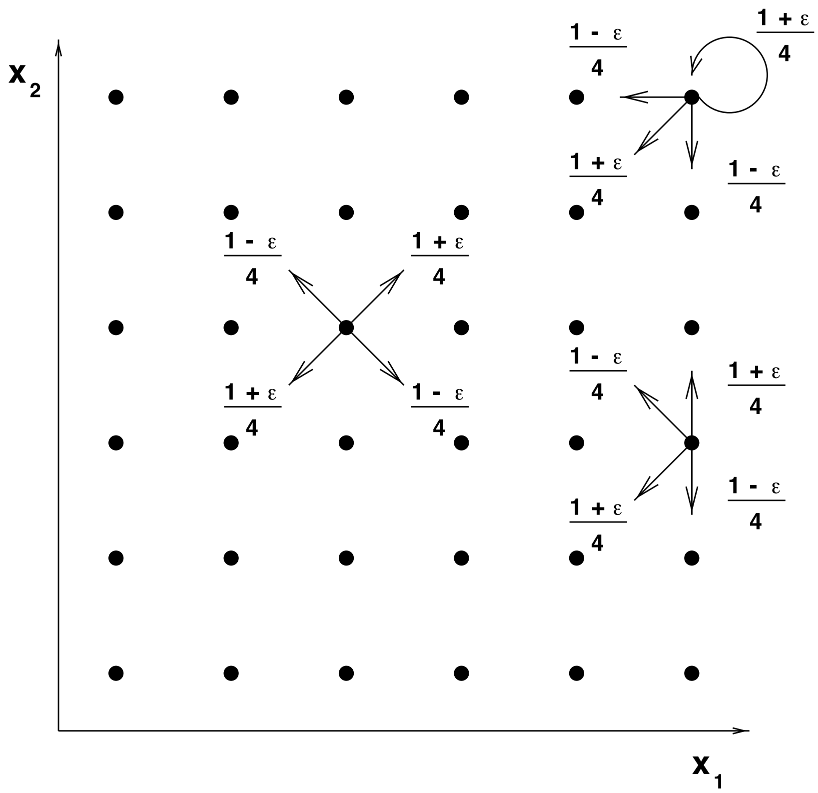

Figure 2. Transition rates for the master equation in a infinite potential well. The three illustrated points are bulk, boundary and corner sites, as shown.

also tends to a δ-function. Here we have plotted $P(d)$ for even values of d only, as the dynamics described above introduces two sub-lattices coupled through the boundary conditions.

A continuum description of the problem can be obtained from the small a expansion of the master equation. By Taylor expanding equations (18)–(20) up to the first non-vanishing terms in a and Δt , and imposing diffusive scaling $\Delta t = a^2/2$ in the limit $a \to 0$, we obtain the continuum equation

$$\frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial^2 x_1} + \frac{\partial^2 P}{\partial^2 x_2} + 2\epsilon \frac{\partial^2 P}{\partial x_1 \partial x_2} \tag{21}$$

with boundary conditions

$$\frac{\partial P}{\partial x_1} + 2\epsilon \frac{\partial P}{\partial x_2} = 0 \tag{22}$$

$$\frac{\partial P}{\partial x_2} + 2\epsilon \frac{\partial P}{\partial x_1} = 0 \tag{23}$$

along $x_1 = 0, L$ and $x_2 = 0, L$, respectively. Note that these equations do not correspond to conventional reflective boundary conditions. In fact, they introduce a probability current into the system which flows from the lighter to the darker regions of figure 3. Only for $\epsilon = 0$ is the probability flux in the steady-state zero, as only in this limit is detailed balance recovered.

At the corners $(0, 0)$ and (L, L), the expansion to lowest order gives

$$P = (1 + \epsilon)P \tag{24}$$

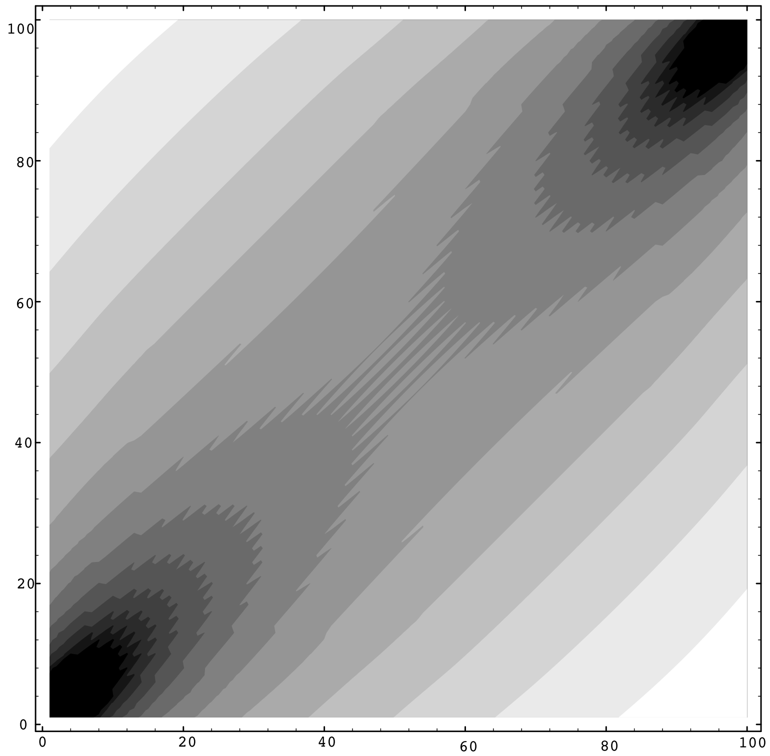

Figure 3. Density plot for the stationary probability distribution of equation (17). The system is an infinite potential well ($L = 100$, $\epsilon = 0.8$) and transitions rate are reported in figure 2. Darker regions correspond to higher values of probability.

whilst at the other two corners

$$P = (1 - \epsilon)P. \tag{25}$$

For $\epsilon = 0$ the equations are satisfied for all P, whereas for $\epsilon > 0$ these conditions imply that, at the corners, $P \to 0$ in the continuum limit. In this limit, however, one is interested in the probability density $p = P/a^2$, where a is the lattice spacing. From numerical simulations of increasing system sizes we find that, as $a \to 0$, $p(0, L) = p(L, 0) \to 0$ whereas $p(0, 0) = p(L, L) \to \infty$. This behaviour shows that, in the continuum limit, the limit $\epsilon \to 0$ might be singular and suggests that any systematic attempt to perturb about the stationary solution for the system with uncorrelated noises is likely to fail.

We turn now to a different class of problems, namely, diffusion in disordered systems. There is considerable current interest in the transport properties of particles in random environments [17]. One usually considers a single Brownian particle governed by equation (1) in which $V(x)$ is a quenched random potential. The dynamics can be defined on a lattice, and the hopping rates between neighbouring sites are chosen to be of the form

$$W_{i\to j} \sim \exp(-\beta(V_j - V_i)) \tag{26}$$

where V_i is the potential on site i and β is an inverse-temperature parameter. Two simple choices can be made for the quenched potential V_i. If the potential is assumed to be bounded on long-length scales, i.e. $V_i = \eta_i$ where η_i is uncorrelated white noise, the late-time dynamics is diffusive with a typical displacement $x \sim t^{1/2}$. However, if the potential is itself a random

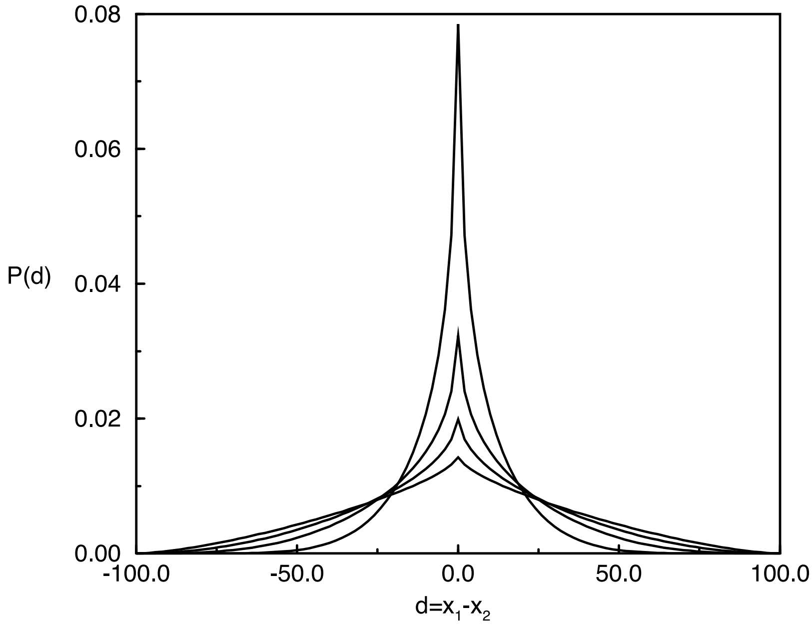

Figure 4. Probability distribution $P(d)$ in the case of infinite potential well ($L = 100$). The curves correspond to $\epsilon = 0.3, 0.5, 0.7, 0.9$ with the larger values of ϵ having greater weight at the origin.

walk, $V_i = V_{i-1} + \eta_i$, then the particle's dynamics is extremely slow, with $x \sim \log^2(t)$ in the long-time limit [14]. This is because, in the case of a rough potential, the effective potential barriers grow on increasing length scales.

Here we are interested in the dynamics of two Brownian particles, moving in a disordered environment, and subjected to correlated noises. In particular, we will restrict our study to the case of completely correlated noise, $\epsilon = 1$, which corresponds to the two particles experiencing the same noise source. In the absence of any potential the two particles would never meet because their relative separation will remain a constant, whereas in the confining potentials discussed above, this limit results in complete aggregation. Will similar behaviour be present for motion in a random potential, even if it remains smooth on long-length scales?

In order to address this issue we will consider the following question: given that if the two particles meet they will remain together forever, what is the distribution of meeting times for two particles subject to the same noise and in the same random environment? First passage problems of this type have been studied in the context of persistence phenomena [15] and aggregation in driven granular gases [16]. From knowledge of the asymptotic behaviour of the meeting time distribution, one can infer the probability of meeting in the long-time limit. In figure 5 we show the meeting time distributions for two particles moving in a rough potential, with both uncorrelated and correlated noise. The noise correlation is introduced into the simulations by using the same random number for each particle, in conjunction with the spatially dependent transition rate, equation (26). The random numbers used to define the potential, η_i, were drawn from a Gaussian distribution with unit variance and we took $\beta = 1$. The initial particle separation, combined with the transition rate, merely sets the timescale for the first encounter and does not change the asymptotic scaling behaviour. In the long-time limit the two distributions appear to be the same, decaying as $1/(t \ln^4 t)$, in agreement with recent analytical calculations [17]. Thus, even with correlated noises the two particles will meet and

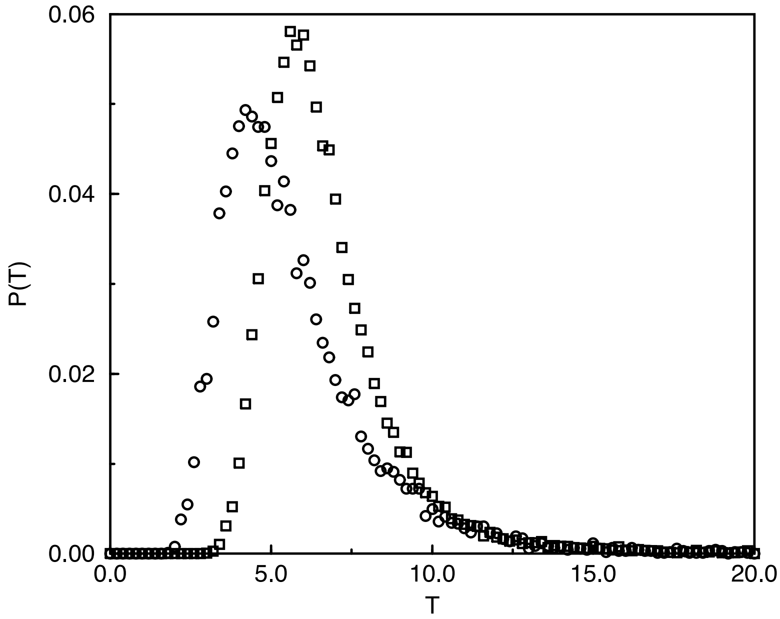

Figure 5. Meeting time distributions in logarithmic time $T = \ln t$ for two particles in a rough potential, with correlated (□) and uncorrelated (○) noise. The initial separation between particles is ten lattice spacings.

then remain together. The equivalence of the two distributions as $t \to \infty$ also suggests that the noise correlation is irrelevant in the long-time limit. This may be due to the existence of growing potential barriers on long-length scales which dominate the late-time dynamics.

In figure 6 we show the meeting time distribution for two particles moving in a smooth random potential. In both cases, with uncorrelated or completely correlated noises ($\epsilon = 1$), we find that the distribution decays asymptotically as $1/t^{3/2}$. Note that this is the same behaviour as that of the single-particle first-return distribution for Brownian motion, which has the same asymptotic form as the meeting time distribution of two independent random walkers. In addition, we observe that the two curves are related by a simple rescaling of time. One can thus interpret the effect of the noise correlation as a rescaling of the effective diffusion coefficient for the two-particle motion. Consequently, even in the presence of a smooth, non-confining potential, the two particles will coalesce and remain together ever after.

Finally, we consider the motion of two particles whose noise correlator depends on the relative distance so that, in (8), $\epsilon = \epsilon(x_1 - x_2)$. We require that $\epsilon(0) = 1$ (perfectly correlated noise) and $\epsilon(\infty) = 0$ (completely uncorrelated noise), which is appropriate, for instance, in the case of two particles immersed in a fluid. The first condition implies that, once the particles are at the same point at the same time, they will remain together so that $P(x_1, x_2) \sim \delta(x_1 - x_2)\mathrm{e}^{-V(x_1)/D}$ is a possible solution for the stationary probability distribution. However, it is not *a priori* obvious that the two particles will always be able to meet under generic conditions. In order to elucidate this point we consider two particles in a harmonic external potential. In the variable $d = x_1 - x_2$ the equation of motion is

$$\dot{d} = -kd + \sqrt{\tilde{D}(d)}\xi(t) \tag{27}$$

where $\tilde{D}(d) = 2D(1 - \epsilon(d))$ and $\langle \xi(t)\xi(t') \rangle = 2\delta(t - t')$. The corresponding Fokker–Planck

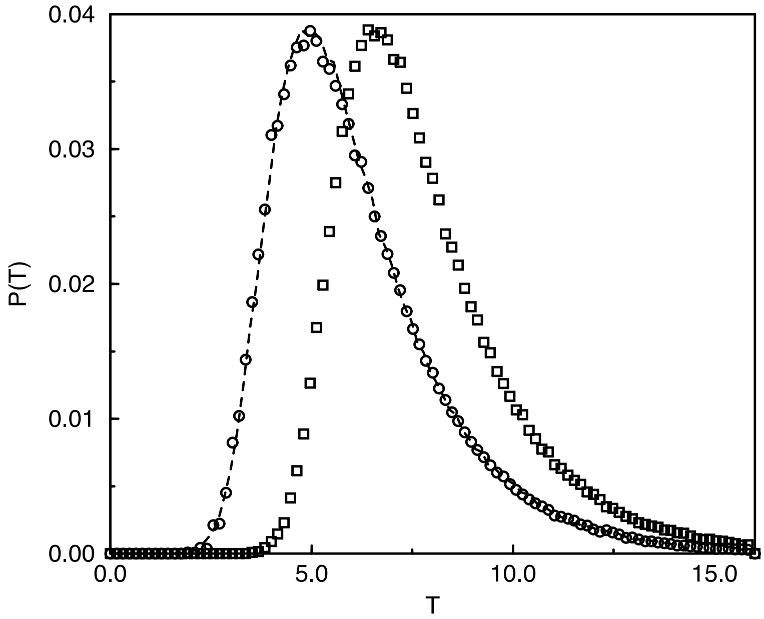

Figure 6. Meeting-time distributions in logarithmic time $T = \ln t$ for two particles in a smooth potential, with correlated (□) and uncorrelated (○) noise. The starting particle separation is 10. The continuous line shows the correlated-noise data shifted by $\Delta T = 1.65$, which corresponds to a simple rescaling of t.

equation, in the Ito convention [18, 19], is

$$\frac{\partial P(d,t)}{\partial t} = \frac{\partial}{\partial d}\left[kdP(d,t) + \frac{\partial(\tilde{D}(d)P(d,t))}{\partial d}\right] \tag{28}$$

and the stationary probability distribution can be formally written as

$$P_{\mathrm{st}}(d) \propto \frac{1}{\tilde{D}(d)} \mathrm{e}^{-k\int_{d_0}^{d} \frac{y}{\tilde{D}(y)}\mathrm{d}y}. \tag{29}$$

Without loss of generality we can take $\tilde{D}(d) \sim d^{\alpha}$ for small d, where α is a suitable parameter. It is clear that the distribution (29) is an acceptable solution only for $\alpha < 1$. In the limit $\alpha \to 1$, the distribution (29) tends to $P_{\mathrm{st}}(d) \sim \delta(d)$, which is also the solution for $\alpha \geqslant 1$ (in fact the distribution (29) with $\alpha \geqslant 1$ is not normalizable). Intuitively, this different behaviour as a function of α can be understood by means of the following heuristic argument. If $\alpha > 1$, $\epsilon'(0) = 0$ so that, for d sufficiently small, the noises on the two particles are 'almost' perfectly correlated and the external potential makes them coalesce. On the other hand, if $\alpha < 1$ $\epsilon'(0) = -\infty$, so that even for small d the noises are sufficiently uncorrelated to keep them apart.

In summary, we have discussed the evolution of two particles driven by correlated noise sources, in the presence of different types of external potentials. We find that, in the presence of a confining potential, particles tend to aggregate as the correlation in the noises is increased and the trajectories converge in the case of identical noises. This effect arises from the combined action of the correlated noises and the confining external potential, as clearly emerges in the example of the infinite potential well. We have limited ourselves to a one-dimensional motion as we do not expect the dimensionality of the space to be relevant in this case. We observe that, if noises are perfectly correlated, particles coalesce even in the presence of a non-confining disordered potential. In this case, however, the one-dimensional character of the model might be important, as particles moving in a three-dimensional environment are expected, with a finite probability, not to meet. Our investigation extends in many respects previous studies, both in the choice of the correlation in the noises and in the form of the external potential. One obvious generalization of our findings is to the case of N particles where correlated noise induces a macroscopic aggregation. This property can be useful in studying phase transitions from a dynamical point of view [20].

Acknowledgments

We warmly acknowledge fruitful discussions with Jayanth Banavar and Maria Pia Riva. This work has been supported by INFM and INFN. SL thanks EPSRC for financial support.

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